# Lecture 10

# Spin Angular Momentum, Complex Poynting's Theorem, Lossless Condition, Energy Density

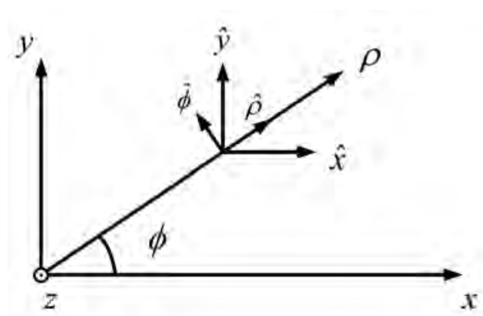


Figure 10.1: The local coordinates used to describe a circularly polarized wave: In cartesian and polar coordinates.

## 10.1 Spin Angular Momentum and Cylindrical Vector Beam

In this section, we will study the spin angular momentum of a circularly polarized (CP) wave. It is to be noted that in cylindrical coordinates, as shown in Figure 10.1, $\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$,

$\hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$, then a CP field is proportional to

$$(\hat{x} \pm j\hat{y}) = \hat{\rho}e^{\pm j\phi} \pm j\hat{\phi}e^{\pm j\phi} = e^{\pm j\phi}(\hat{\rho} \pm \hat{\phi}) \tag{10.1.1}$$

Therefore, the $\hat{\rho}$ and $\hat{\phi}$ of a CP is also an azimuthal traveling wave in the $\hat{\phi}$ direction in addition to being a traveling wave $e^{-j\beta z}$ in the $\hat{z}$ direction. This is obviated by writing

$$e^{-j\phi} = e^{-jk_\phi \rho\phi} \tag{10.1.2}$$

where $k_\phi = 1/\rho$ is the azimuthal wave number, and $\rho\phi$ is the arc length traversed by the azimuthal wave. Notice that the wavenumber $k_\phi$ is dependent on $\rho$: the larger the $\rho$, the larger the azimuthal wavelength. Thus, the wave possesses angular momentum called the spin angular momentum (SAM), just as a traveling wave $e^{-j\beta z}$ possesses linear angular momentum in the $\hat{z}$ direction.

In optics research, the generation of cylindrical vector beam is in vogue. Figure 10.2 shows a method to generate such a beam. A CP light passes through a radial analyzer that will only allow the radial component of (10.1.1) to be transmitted. Then a spiral phase element (SPE) compensates for the $\exp(\pm j\phi)$ phase shift in the azimuthal direction. Finally, the light is a cylindrical vector beam which is radially polarized without spin angular momentum. Such a beam has been found to have nice focussing property, and hence, has aroused researchers' interest in the optics community [74].

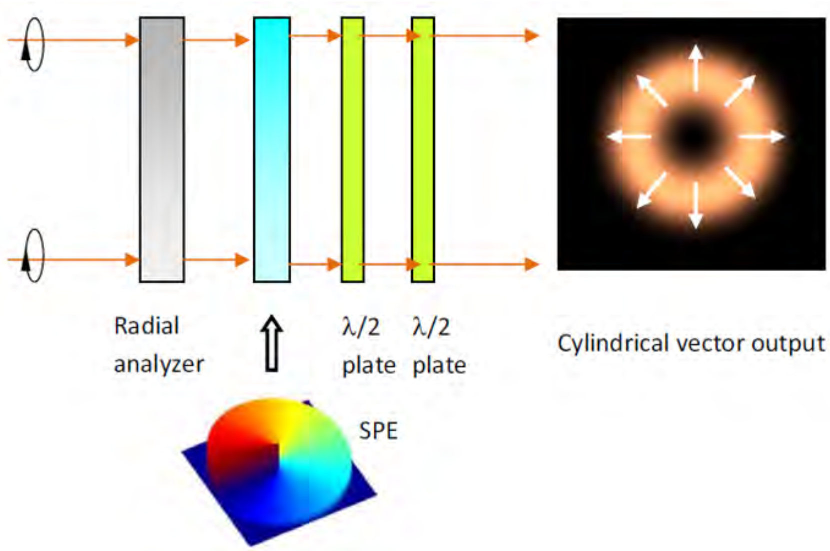


Figure 10.2: A cylindrical vector beam can be generated experimentally. The spiral phase element (SPE) compensates for the $\exp(\pm j\phi)$ phase shift (courtesy of Zhan, Q. [74]).

## 10.2 Complex Poynting's Theorem and Lossless Conditions

### 10.2.1 Complex Poynting's Theorem

It has been previously shown that the vector $\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$ has a dimension of watts/m$^2$ which is that of power density. Therefore, it is associated with the direction of power flow [31, 43]. As has been shown for time-harmonic field, a time average of this vector can be defined as

$$\langle\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)\,dt. \tag{10.2.1}$$

Given the phasors of time harmonic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$, namely, $\mathbf{E}(\mathbf{r},\omega)$ and $\mathbf{H}(\mathbf{r},\omega)$ respectively, we can show that

$$\langle\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)\rangle = \frac{1}{2}\,\Re e\{\mathbf{E}(\mathbf{r},\omega)\times\mathbf{H}^*(\mathbf{r},\omega)\}. \tag{10.2.2}$$

Here, the vector $\mathbf{E}(\mathbf{r},\omega)\times\mathbf{H}^*(\mathbf{r},\omega)$, as previously discussed, is also known as the complex Poynting vector. Moreover, because of its aforementioned property, and its dimension of power density, we will study its conservative property. To do so, we take its divergence and use the appropriate vector identity to obtain[1]

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) = \mathbf{H}^*\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H}^*. \tag{10.2.3}$$

Next, using Maxwell's equations for $\nabla\times\mathbf{E}$ and $\nabla\times\mathbf{H}^*$, namely

$$\nabla\times\mathbf{E} = -j\omega\mathbf{B} \tag{10.2.4}$$

$$\nabla\times\mathbf{H}^* = -j\omega\mathbf{D}^* + \mathbf{J}^* \tag{10.2.5}$$

and the constitutive relations for anisotropic media that

$$\mathbf{B} = \overline{\boldsymbol{\mu}}\cdot\mathbf{H}, \quad \mathbf{D}^* = \overline{\varepsilon}^*\cdot\mathbf{E}^* \tag{10.2.6}$$

we have

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) = -j\omega\,\mathbf{H}^*\cdot\mathbf{B} + j\omega\,\mathbf{E}\cdot\mathbf{D}^* - \mathbf{E}\cdot\mathbf{J}^* \tag{10.2.7}$$

$$= -j\omega\,\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\mathbf{H} + j\omega\,\mathbf{E}\cdot\overline{\varepsilon}^*\cdot\mathbf{E}^* - \mathbf{E}\cdot\mathbf{J}^*. \tag{10.2.8}$$

The above is also known as the complex Poynting's theorem. It can also be written in an integral form using Gauss' divergence theorem, namely,

$$\int_S d\mathbf{S}\cdot(\mathbf{E}\times\mathbf{H}^*) = -j\omega\int_V dV(\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\mathbf{H} - \mathbf{E}\cdot\overline{\varepsilon}^*\cdot\mathbf{E}^*) - \int_V dV\mathbf{E}\cdot\mathbf{J}^*. \tag{10.2.9}$$

where $S$ is the surface bounding the volume $V$.

[1]We will drop the argument $\mathbf{r},\omega$ for the phasors in our discussion next as they will be implied.

### 10.2.2 Lossless Conditions

For a region $V$ that is lossless and source-free, $\mathbf{J} = 0$. There should be no net time-averaged power-flow out of or into this region $V$. Therefore,

$$\Re e \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = 0, \tag{10.2.10}$$

Because of energy conservation, the real part of the right-hand side of (10.2.8), without the $\mathbf{E} \cdot \mathbf{J}^*$ term, must be zero. In other words, the right-hand side of (10.2.8) should be purely imaginary. Thus

$$\int_V dV (\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \mathbf{E}^*) \tag{10.2.11}$$

must be a real quantity.

Other than the possibility that the above is zero, the general requirement for (10.2.11) to be real for arbitrary $\mathbf{E}$ and $\mathbf{H}$, is that $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ and $\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \mathbf{E}^*$ are real quantities. This is only possible if $\overline{\boldsymbol{\mu}}$ is hermitian.[2] Therefore, the conditions for anisotropic media to be lossless are

$$\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^\dagger, \qquad \overline{\boldsymbol{\varepsilon}} = \overline{\boldsymbol{\varepsilon}}^\dagger, \tag{10.2.12}$$

requiring the permittivity and permeability tensors to be hermitian. If this is the case, (10.2.11) is always real for arbitraty $\mathbf{E}$ and $\mathbf{H}$, and (10.2.10) is true, implying a lossless region $V$. Notice that for an isotropic medium, this lossless conditions reduce simply to that $\Im m(\mu) = 0$ and $\Im m(\varepsilon) = 0$, or that $\mu$ and $\varepsilon$ are pure real quantities. Looking back, many of the effective permittivities or dielectric constants that we have derived using the Drude-Lorentz-Sommerfeld model cannot be lossless when the friction term is involved.

If a medium is source-free, but lossy, then $\Re e \int d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) < 0$. In other words, time-average power must flow inward to the volume $V$. Consequently, from (10.2.9) without the source term $\mathbf{J}$, this implies

$$\Im m \int_V dV (\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \mathbf{E}^*) < 0. \tag{10.2.13}$$

But the above, using that $\Im m(Z) = 1/(2j)(Z - Z^*)$ and that $(\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H})^\dagger = \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}}^\dagger \cdot \mathbf{H}$, is the same as

$$-j \int_V dV [\mathbf{H}^* \cdot (\overline{\boldsymbol{\mu}}^\dagger - \overline{\boldsymbol{\mu}}) \cdot \mathbf{H} + \mathbf{E}^* \cdot (\overline{\boldsymbol{\varepsilon}}^\dagger - \overline{\boldsymbol{\varepsilon}}) \cdot \mathbf{E}] > 0. \tag{10.2.14}$$

[2] $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ is real only if its complex conjugate, or conjugate transpose is itself. Using some details from matrix algebra that $(\overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \cdot \overline{\mathbf{C}})^t = \overline{\mathbf{C}}^t \cdot \overline{\mathbf{B}}^t \cdot \overline{\mathbf{A}}^t$, implies that (in physics notation, the transpose of a vector is implied in a dot product) $(\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H})^\dagger = (\mathbf{H} \cdot \overline{\boldsymbol{\mu}}^* \cdot \mathbf{H}^*)^t = \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}}^\dagger \cdot \mathbf{H} = \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$. The last equality in the above is possible only if $\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^\dagger$ or that $\overline{\boldsymbol{\mu}}$ is hermitian.

Therefore, for a medium to be lossy, $-j(\overline{\boldsymbol{\mu}}^\dagger - \overline{\boldsymbol{\mu}})$ and $-j(\overline{\varepsilon}^\dagger - \overline{\varepsilon})$ must be hermitian, positive definite matrices, to ensure the inequality in (10.2.14). Similarly, for an active medium, $-j(\overline{\boldsymbol{\mu}}^\dagger - \overline{\boldsymbol{\mu}})$ and $-j(\overline{\varepsilon}^\dagger - \overline{\varepsilon})$ must be hermitian, negative definite matrices.

For a lossy medium which is conductive, we may define $\mathbf{J} = \overline{\boldsymbol{\sigma}} \cdot \mathbf{E}$ where $\overline{\boldsymbol{\sigma}}$ is a general conductivity tensor. In this case, equation (10.2.9), after combining the last two terms, may be written as

$$\int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = -j\omega \int_V dV \left[\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \left(\overline{\varepsilon}^* + \frac{j\overline{\boldsymbol{\sigma}}^*}{\omega}\right) \cdot \mathbf{E}^*\right] \tag{10.2.15}$$

$$= -j\omega \int dV [\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \tilde{\overline{\varepsilon}}^* \cdot \mathbf{E}^*], \tag{10.2.16}$$

where $\tilde{\overline{\varepsilon}} = \overline{\varepsilon} - \frac{j\overline{\boldsymbol{\sigma}}}{\omega}$ which is the general complex permittivity tensor. In this manner, (10.2.16) has the same structure as the source-free Poynting's theorem. Notice here that the complex permittivity tensor $\tilde{\overline{\varepsilon}}$ is clearly non-hermitian corresponding to a lossy medium.

For a lossless medium without the source term, by taking the imaginary part of (10.2.9), we arrive at

$$\Im m \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = -\omega \int_V dV (\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^*), \tag{10.2.17}$$

The left-hand side of the above is the reactive power coming out of the volume $V$, and hence, the right-hand side can be interpreted as reactive power as well. It is to be noted that $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ and $\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^*$ are not to be interpreted as stored energy density when the medium is dispersive. The correct expressions for stored energy density will be derived in the next section.

But, the quantity $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ for lossless, dispersionless media is associated with the time-averaged energy density stored in the magnetic field, while the quantity $\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^*$ for lossless dispersionless media is associated with the time-averaged energy density stored in the electric field. Then, for lossless, dispersionless, source-free media, then the right-hand side of the above can be interpreted as stored energy density. Hence, the reactive power is proportional to the time rate of change of the difference of the time-averaged energy stored in the magnetic field and the electric field.

## 10.3 Energy Density in Dispersive Media

A dispersive medium alters our concept of what energy density is.[3] To this end, we assume that the field has complex $\omega$ dependence in $e^{j\omega t}$, where $\omega = \omega' - j\omega''$, rather than real $\omega$ dependence. We take the divergence of the complex power for fields with such time dependence, and let $e^{j\omega t}$ be attached to the field. So $\mathbf{E}(t)$ and $\mathbf{H}(t)$ are complex field but not exactly like phasors since they are not truly time harmonic. In other words, we let

$$\mathbf{E}(\mathbf{r}, t) = \underset{\sim}{\mathbf{E}}(\mathbf{r}, \omega) e^{j\omega t}, \quad \mathbf{H}(\mathbf{r}, t) = \underset{\sim}{\mathbf{H}}(\mathbf{r}, \omega) e^{j\omega t} \tag{10.3.1}$$

[3]The derivation here is inspired by H.A. Haus, Electromagnetic Noise and Quantum Optical Measurements [75]. Generalization to anisotropic media is given by W.C. Chew, Lectures on Theory of Microwave and Optical Waveguides [76].

The above, just like phasors, can be made to satisfy Maxwell's equations where the time derivative becomes $j\omega$ but with complex $\omega$. We can study the quantity $\mathbf{E}(\mathbf{r},t)\times\mathbf{H}^*(\mathbf{r},t)$ which has the unit of power density. In the real $\omega$ case, their time dependence will exactly cancel each other and this quantity becomes complex power again, but not in the complex $\omega$ case. Hence,

$$\begin{aligned}\nabla\cdot[\mathbf{E}(t)\times\mathbf{H}^*(t)] &= \mathbf{H}^*(t)\cdot\nabla\times\mathbf{E}(t)-\mathbf{E}(t)\cdot\nabla\times\mathbf{H}^*(t)\\ &= -\mathbf{H}^*(t)\cdot j\omega\mu\mathbf{H}(t)+\mathbf{E}(t)\cdot j\omega^*\varepsilon^*\mathbf{E}^*(t)\end{aligned}\tag{10.3.2}$$

where Maxwell's equations have been used to substitute for $\nabla\times\mathbf{E}(t)$ and $\nabla\times\mathbf{H}^*(t)$. The space dependence of the field is implied, and we assure a source-free medium so that $\mathbf{J}=0$.

If $\mathbf{E}(t)\sim e^{j\omega t}$, then, due to $\omega$ being complex, now $\mathbf{H}^*(t)\sim e^{-j\omega^* t}$, and the term like $\mathbf{E}(t)\times\mathbf{H}^*(t)$ is not truly time independent but becomes

$$\mathbf{E}(t)\times\mathbf{H}^*(t)\sim e^{j(\omega-\omega^*)t}=e^{2\omega'' t}\tag{10.3.3}$$

And each of the term above will have similar time dependence. Writing (10.3.2) more explicitly, by letting $\omega=\omega'-j\omega''$, we have

$$\nabla\cdot[\mathbf{E}(t)\times\mathbf{H}^*(t)]=-j(\omega'-j\omega'')\mu(\omega)|\mathbf{H}(t)|^2+j(\omega'+j\omega'')\varepsilon^*(\omega)|\mathbf{E}(t)|^2\tag{10.3.4}$$

Assuming that $\omega''\ll\omega'$, or that the field is quasi-time-harmonic, we can let, after using Taylor series approximation, that

$$\mu(\omega'-j\omega'')\cong\mu(\omega')-j\omega''\frac{\partial\mu(\omega')}{\partial\omega'},\quad \varepsilon(\omega'-j\omega'')\cong\varepsilon(\omega')-j\omega''\frac{\partial\varepsilon(\omega')}{\partial\omega'}\tag{10.3.5}$$

Using (10.3.5) in (10.3.4), and collecting terms of the same order, and ignoring $(\omega'')^2$ terms, gives[4]

$$\begin{aligned}\nabla\cdot[\mathbf{E}(t)\times\mathbf{H}^*(t)]\cong&-j\omega'\mu(\omega')|\mathbf{H}(t)|^2+j\omega'\varepsilon^*(\omega')|\mathbf{E}(t)|^2\\ &-\omega''\mu(\omega')|\mathbf{H}(t)|^2-\omega'\omega''\frac{\partial\mu}{\partial\omega'}|\mathbf{H}(t)|^2\\ &-\omega''\varepsilon^*(\omega')|\mathbf{E}(t)|^2-\omega'\omega''\frac{\partial\varepsilon^*}{\partial\omega'}|\mathbf{E}(t)|^2\end{aligned}\tag{10.3.6}$$

The above can be rewritten as

$$\begin{aligned}\nabla\cdot[\mathbf{E}(t)\times\mathbf{H}^*(t)]\cong&-j\omega'\left[\mu(\omega')|\mathbf{H}(t)|^2-\varepsilon^*(\omega')|\mathbf{E}(t)|^2\right]\\ &-\omega''\left[\frac{\partial\omega'\mu(\omega')}{\partial\omega'}|\mathbf{H}(t)|^2+\frac{\partial\omega'\varepsilon^*(\omega')}{\partial\omega'}|\mathbf{E}(t)|^2\right]\end{aligned}\tag{10.3.7}$$

The above approximation is extremely good when $\omega''\ll\omega'$. For a lossless medium, $\varepsilon(\omega')$ and $\mu(\omega')$ are purely real, and the first term of the right-hand side is purely imaginary while the

[4] This is the general technique of perturbation expansion [40].

second term is purely real. In the limit when $\omega'' \to 0$, when we take half the imaginary part of the above equation, we have

$$\nabla \cdot \frac{1}{2}\Im m\left[\mathbf{E}\times\mathbf{H}^*\right] = -\omega'\left[\frac{1}{2}\mu|\mathbf{H}|^2 - \frac{1}{2}\varepsilon|\mathbf{E}|^2\right] \tag{10.3.8}$$

which has the physical interpretation of reactive power as has been previously discussed. When we take half the real part of (10.3.7), we obtain

$$\nabla \cdot \frac{1}{2}\Re e[\mathbf{E}\times\mathbf{H}^*] = -\frac{\omega''}{2}\left[\frac{\partial\omega'\mu}{\partial\omega'}|\mathbf{H}|^2 + \frac{\partial\omega'\varepsilon}{\partial\omega'}|\mathbf{E}|^2\right] \tag{10.3.9}$$

Since the right-hand side has time dependence of $e^{2\omega'' t}$, it can be written as

$$\nabla \cdot \frac{1}{2}\Re e[\mathbf{E}\times\mathbf{H}^*] = -\frac{\partial}{\partial t}\frac{1}{4}\left[\frac{\partial\omega'\mu}{\partial\omega'}|\mathbf{H}|^2 + \frac{\partial\omega'\varepsilon}{\partial\omega'}|\mathbf{E}|^2\right] = -\frac{\partial}{\partial t}\langle W_T\rangle \tag{10.3.10}$$

Therefore, the time-average stored energy density can be identified as

$$\langle W_T\rangle = \frac{1}{4}\left[\frac{\partial\omega'\mu}{\partial\omega'}|\mathbf{H}|^2 + \frac{\partial\omega'\varepsilon}{\partial\omega'}|\mathbf{E}|^2\right] \tag{10.3.11}$$

For a non-dispersive medium, the above reverts to

$$\langle W_T\rangle = \frac{1}{4}\left[\mu|\mathbf{H}|^2 + \varepsilon|\mathbf{E}|^2\right] \tag{10.3.12}$$

which is what we have derived before.

In the above analysis, we have used a quasi-time-harmonic signal with $\exp(j\omega t)$ dependence. In the limit when $\omega'' \to 0$, this signal reverts back to a time-harmonic signal, and to our usual interpretation of complex power. However, by assuming the frequency $\omega$ to have a small imaginary part $\omega''$, it forces the stored energy to grow very slightly, and hence, power has to be supplied to maintain the growth of this stored energy. By so doing, it allows us to identify the expression for energy density for a dispersive medium. These expressions for energy density were not discovered until 1960 by Brillouin [77], as energy density times group velocity should be power flow. More discussion on this topic can be found in Jackson [43].

It is to be noted that if the same analysis is used to study the energy storage in a capacitor or an inductor, the energy storage formulas have to be accordingly modified if the capacitor or inductor is frequency dependent.